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THE CYCLE LENGTH OF SPARSE REGULAR GRAPH

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Abstract. Let G be a d −reguler graph with girth g. Set of cycle length in Graf G is denoted by $C(G)$. Graph G is a sparse graph if and only if $\frac{2.|E|}{|V|(|V|-1)} < \frac{1}{2}$. Furthermore, it was obtained the number of cycle length of sparse d −reguler graph which denoted $|C(G)|$ is $\Omega\left(d^{\lfloor (g-1)/2 \rfloor}\right)$.

INTRODUCTION

Let $C(G)$ denote the set of lengths of cycles in a graph G. According to (Erdös, 1993), every graph with minimum degree three contains a cycle with length 2^n, which n is the number of vertices in G. An advance research was initiated by (Erdős et al., 1999), it was founded that the lower bound of cycle lengths of graph with minimum degree k and girth g is $ck^{g/8}$. The recent research was done by (Groenland et al., 2022; Sudakov & Verstraëte, 2007) which shows that sparse graph with average degree d and girth g contains cycle with lengths $\Omega\left(d^{\lfloor (g-1)/2 \rfloor}\right)$.

Since n −vertex graphs with average degree d may have girth at least $\log_{d-1} n$, we cannot guarantee $C(G)$ for sparse graph contains integer from a finite set. (Erdos & Hajnal, 1969) conjectured

$$\sum_{l \in C(G)} \frac{1}{l} = \Omega(\log d)$$

whenever G has average degree d. (Here and throughout the paper the notation $a_d = \Omega(b_d)$ means that there is an absolute constant C such that $a_d \geq C b_d$ when $d \to \infty$). Therefore, if a graph does not have too many short cycles, then it must have many long cycles. Thus, the aim of this paper is to find the lower bound of $|C(G)|$ when G is a d −reguler graph with girth g. The condition of girth G will closely related to the condition of its sparsity.

LITERATURE REVIEW

PRELIMINARIES

Let G be a d −reguler graph with girth g. $C(G)$ is the set of cycle length in graph G. The example of regular graph is Moore Graph (Bannai & Ito, 1973). Moore Graph is a graph with minimum degree d and girth g. The number of vertices in Moore Graph states

$$|V(G)| \geq \begin{cases} 1 + d + d(d-1) + \cdots + d(d-1)^{\left\lfloor \frac{g-1}{2} \right\rfloor - 1} & if\ g\ is\ odd \\ 2\left(1 + d + d(d-1) + \cdots + d(d-1)^{\left\lfloor \frac{g-1}{2} \right\rfloor - 1}\right) & if\ g\ is\ even \end{cases}$$

An open neighborhood of $X \subset V(G)$ in graph G is defined by

$$\partial X = \{y \in V(G) \backslash X \mid \exists x \in X : \{x, y\} \in E(G)\}$$

The open neighborhood of X is a set of vertices which is not in X and adjacent to at least one vertex of X. The d −core of graph G (if exists), is a subgraph which obtained by omitting vertices which degree is $d - 1$. Thus, if a graph has average degree $2d$, then it has a d −core.

Theorem 1. Let G be a −reguler graph with girth g. Then $C(G)$ contains $\Omega(d^{\lfloor (g-1)/2 \rfloor})$ consecutive even integers.

The aim of this paper is to prove Theorem 1 by using these following Lemma and Theorem.

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RESULT

Lemma 2. Let G is $6(d+1)$ −regular graph with girth g, then for every $X \subset V(G)$ with maximum size $\frac{1}{3}d^{\lceil (g-1)/2 \rceil}$,

$$|\partial X| > 2|X|$$

Proof. Suppose $|\partial X| \leq 2|X|$ for some $X \subset V(G)$. We will show that $|X| > \frac{1}{3}d^{\lceil (g-1)/2 \rceil}$. Let H be a subgraph of G which spanned by $Y = X \cup \partial X$. Because ∂X is a set of vertices which are not in X, therefore ∂X and X are mutually exclusive. Thus,

$$|Y| = |X| + |\partial X|$$
$$|\partial X| = |Y| - |X|$$

The first argumentation in proving this lemma, we have $|\partial X| \leq 2|X|$. So, we obtain

$$|Y| - |X| \leq 2|X|$$
$$|Y| \leq 3|X|$$

From Theorem of the number of edge related to the degree of a graph, we obtain

$$e(H) \geq \frac{1}{2}\sum_{x \in X} d(x) \geq \frac{1}{2}6(d+1)|X| \geq 3(d+1)|X|$$

By substituting

$$|Y| \leq 3|X|$$
$$\frac{1}{3}|Y| \leq |X|$$

We obtain

$$e(H) \geq \frac{1}{2}\sum_{x \in X} d(x) \geq \frac{1}{2}6(d+1)|X| \geq 3(d+1)|X| \geq (d+1)|Y|$$

Thus, H contain a subgraph Γ with minimum degree $d+1$.
By applying *Moore Bound* (Alon et al., 2002), we obtain

$$|Y| \leq 3|X|$$
$$3|X| \geq |Y| \geq |V(\Gamma)| > 1 + \sum_{i=0}^{k-1} d(d-1)^i$$
$$3|X| \geq |Y| \geq |V(\Gamma)| > 1 + d\sum_{i=0}^{k-1} (d-1)^i$$

Because the minimum degree is $d+1$, by substituting $d+1$ we obtain

$$3|X| \geq |Y| \geq |V(\Gamma)| > 1 + (d+1) \sum_{i < \left\lceil \frac{g-1}{2} \right\rceil} (d+1-1)^i$$
$$3|X| \geq |Y| \geq |V(\Gamma)| > 1 + (d+1) \sum_{i < \left\lceil \frac{g-1}{2} \right\rceil} d^i$$
$$3|X| \geq |Y| \geq |V(\Gamma)| > 1 + (d+1) \sum_{i < \left\lfloor \frac{g-1}{2} \right\rfloor} d^i > d^{\lfloor (g-1)/2 \rfloor}$$

Thus

$$3|X| > d^{\lfloor (g-1)/2 \rfloor}$$
$$|X| > \frac{1}{3}d^{\lfloor (g-1)/2 \rfloor}$$

as required.

Theorem 3. For every graph G $48(d+1)$ −regular with girth g, $|C(G)| \geq \frac{1}{8}d^{\lfloor (g-1)/2 \rfloor}$.

Proof. Let H be a maximum bipartite subgraph of G, which contain at least half of the edges of G. Then, some connected component F in H is a graph with average degree at least $24(d+1)$. Let T be a breadth first search tree in F, and let L_i denote the set of vertices of T at distance i from the root of

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T. Since F is bipartite, so there is no edge of F joins two vertices of L_i. $e(L_i, L_{i+1})$ denote the number of edges of F with one endpoint in L_i and one endpoint in L_{i+1}. Accordingly,

$$\sum_i e(L_i, L_{i+1}) = e(F)$$

By Theorem of the number of edge related to the degree of a graph, we obtain

$$\sum_i e(L_i, L_{i+1}) = e(F) \geq \frac{1}{2} 24(d+1)|V(F)|$$

$$\sum_i e(L_i, L_{i+1}) = e(F) \geq 12(d+1)|V(F)|$$

$$\sum_i e(L_i, L_{i+1}) = e(F) \geq 6(d+1)\sum_i |L_i| + |L_{i+1}|$$

Thus, $L_i \cup L_{i+1}$ has average degree at least $12(d+1)$. Then, we obtain subgraph $F_i \subset F$. Then, F_i contain subgraph Γ with average degree $6(d+1)$. By **Lemma 2**, we obtain $|\partial X| > 2|X|$ for every $X \subset V(G)$, has maximum size $\frac{1}{3} d^{\lfloor (g-1)/2 \rfloor}$. By Posa's Lemma (Pósa, 1965; Raymond & Thilikos, 2017), Γ contain path P with length $d^{\lfloor (g-1)/2 \rfloor}$. Let T' be a minimal subtree of T whose set of end vertices is $V(P) \cap L_i$. The minimality of T' ensures that it branches at the root. Let A be the set of vertices in $V(P) \cap L_i$ in one of these branches and let $B = (V(P) \cap L_i) \backslash A$. So, $A, B$are not empty sets and path from A to B through its root have the same length, says $2h$.
We assume,

$$|B| \geq |A|$$

$$|B| \geq \frac{1}{4}|P|$$

$$\frac{1}{2}|B| \geq \frac{1}{8}|P|$$

If a is a vertices in A, therefore, there is exist subpath P from a to a vertices in B of at least $\frac{1}{8}|P|$ different lengths. For any path Q, there is a unique subpath R of T' through the root joining the endpoints of Q, so that $Q \cup R$ is a cycle in G. Since all R have the same length $2h$, we obtain

$$|C(G)| \geq \frac{1}{8} d^{\lfloor (g-1)/2 \rfloor}$$

Lemma 4. Let G be a $48(d+1)-$regular graph with girth g, where $d^{\lfloor (g-1)/2 \rfloor} \geq 6$. Then, G contains $\theta-$graph which contain a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$.

Proof. Let the path P, tree T' and set L_i be defined as in the proof of Theorem 3. Since $d^{\lfloor (g-1)/2 \rfloor} \geq 6$, we have $|V(P) \cap L_i| \geq 3$. Let $Q \subset P$ be a path of length at least $|E(P)| - 2$ with endpoints in L_i. Because $|V(Q) \cap L_i| \geq 3$, therefore Q has an interior vertex in L_i. If R is a path in T' joining the endpoints of Q, then $Q \cup R$ form a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$. So, for some path $S \subset T'$ from the root of T' to an interior vertex of Q in L_i, the subgraph $Q \cup R \cup S$ is the required $\theta-$graph.

It is convenient to define an $AB-$*path* in a graph G to be a path with one endpoint in A and one endpoint in B, where $A, B \subset V(G)$. This following Lemma is obtained by (Bondy & Simonovits, 1974).

Lemma 5. Let Γ be a $\theta-$graph and let (A, B) be a nontrivial partition of $V(\Gamma)$. Then Γ contains $AB-$paths of all lengths less than $|V(\Gamma)|$ unless Γ is bipartite with bipartition (A, B).

Proof of Theorem 1. Let G be a $192(d+1)-$regular graph with girth g and H be a maximum bipartite subgraph of G. Then according to **Theorem 3**, some connected component F of H has average degree at least $96(d+1)$. Let T be a breadth-first search tree in F, and let L_i is the set of vertices of T at distance i from the root. Then, for some i, the subgraph F_i of F induced by

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$L_i \cup L_{i+1}$ has average degree at least $48(d+1)$. By **Lemma 4**, F_i contains a θ −graph Γ which contain a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$. Let T' be the minimal subtree of T whose set of end vertices is $V(\Gamma) \cap L_i$. Then there is a partition (A, B^*) from $V(\Gamma) \cap L_i$, so all AB^* −paths in T' go through the root and have the same length, say $2h$.

Let $B = V(\Gamma)\backslash A$. By **Lemma 5**, there exist AB −paths in Γ of all even lengths in $\{1, 2, \dots, d^{\lfloor (g-1)/2 \rfloor} + 2\}$. Since they have an even length, each such path is actually an AB^* −path, and the union of this path with the unique subpath of T' of length $2h$ joining its endpoints is a cycle. Therefore $C(G)$ contains$d^{\lfloor (g-1)/2 \rfloor}$ consecutive even integers, as required.

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