Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 6, Number 3, July 2022
DOI : https://doi.org/10.33395/sinkron.v7i3.11602
e-ISSN : 2541-2019
p-ISSN : 2541-044X

The Maximum Degree of an Exponentially Distributed Random Graph

Desti Alannora Harahap,[a)] Saib Suwilo, Mardiningsih
University of Sumatera Utara, Medan, Indonesia
destialannorahrp05@gmail.com

Submitted : July 31, 2022 | **Accepted** : June 25, 2022 | **Published** : August 12, 2022

Abstract. Let $G \in \mathcal{G}(n,p)$ be a graph on n vertices where each pair of vertices is joined independently with probability p for $0 < p < 1$ and $q = 1 - p$. In this work, we introduce weighted random graf G with exponential distribution and investigate that the probability that every vertex of G has degree at most $np + b\sqrt{pqn}$ is equal to 0.595656764

Keywords: Exponentially, Random graphic

INTRODUCTION

Let $\mathcal{G}(n,p)$ be the family of all random graph with vertex set $V = [n] = 1,2, \ldots, n$ and $0 < p < 1$ is the probability that each edge is connected independently. For $G \in \mathcal{G}(n,p)$, in this paper we will show the distribution of the maximum degree of random graph G. The maximum degree was started by (Erdos & Rényi, 1960) who studied about asymptotic statistical properties of random graphs for $n \to +\infty$. Furthermore, (Bollobás, 1981) investigated degree sequences $d_1 \geq d_2 \geq \cdots d_n$ of random graph G order n in which the edges are chosen independently and with the same probability p. (McKay & Wormald, 1997) considered that several simple models derived from a set of independent binomial distribution can be accurately approximated the joint distributions of the degrees of a random graph. The study of the distribution of the maximum degree $d^{maks}(G)$ already quite complete discussed by researchers (Bollobás, 1998; Feller, 2008; Riordan & Selby, 2000). However, the results described in above all only applies to a weightless random graph.

The main idea that will be used in this research is weighted random graph. For $G \in \mathcal{G}(n,p)$, where each edge is given an independent random weight, which 1 with probability p and 0 otherwise (Grimmett & Stirzaker, 2020; Kleitman, 1966). The study of the maximum degree of the weighted random graph G which is normally distributed with mean p and variance pq has been studied by (Riordan & Selby, 2000). They showed the probability every vertex of the graph G has the maximum degree. Using the exponential distribution, (Bhamidi et al., 2018) have studied the weighted random graph model for the case of large networks. This research is proposed to model weighted network data emerging from a number of applications including socio-economic data such as migration flows as well as neuroscience.

In this paper, we are interested to show the probability that each vertex of weighted random G with exponential distribution has a maximum degree $pn + b\sqrt{pqn}$.
Our main result is as follows:

THEOREM.
Let $0 < p < 1$, $q = 1 - p$ and b be fixed. For G a random graph from $\mathcal{G}(n,p)$, let

$$p_n = \mathbb{P}\left(d^{maks}(G) \leq np + b\sqrt{npq}\right)$$

*Corresponding author

Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 6, Number 3, July 2022
DOI : https://doi.org/10.33395/sinkron.v7i3.11602
e-ISSN : 2541-2019
p-ISSN : 2541-044X

Then

$$p_n^{1/n} \to c \text{ as } n \to \infty$$

where c is a constant depending only on b and $\lambda > 0$, given by

$$c = maks\{\Phi(b+\theta)e^{-\lambda\theta}\}$$

The above maximum is attained at $\theta_0 = \theta(b)$ the root of equation

$$\theta_0\Phi(b+\theta_0) = \phi(b+\theta_0)$$

PRELIMINARIES

A probability space is a triple (Ω, Σ, P), where Ω is a set, Ω is a field of subsets of Ω, $\mathbb{P}$ is a non-negative measure on Ω and $\mathbb{P}(\Omega) = 1$ (Bollobás, 1981; Courant & Hilbert, 1954). In the simplest case Ω is a finite set and Ω is $\mathbb{P}(\Omega)$, the set of all subsets of Ω. Then $\mathbb{P}$ is determined by the function Ω\rightarrow [0,1], $w \to \mathbb{P}(w)$, namely

$$\mathbb{P}(A) = \sum_{w \in A} \mathbb{P}(w), A \subset \Omega$$

A real valued random variable X is a measurable real-valued function on a probability space, $X: \Omega \to \mathbb{R}$.

Given a real valued X, its distribution function is $F(x) = \mathbb{P}(X < x), -\infty < x < \infty$. Thus $F(x)$ is monotone increasing, continuous from the left, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. If there is a function $f(t)$ such that $F(x) = \int_{-\infty}^{x} f(t)dt$, then $f(t)$ is a density function of X.

A random variable is said to be exponential distributed with mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda}^2$ if it has the exponential density function

$$\phi(x) = \lambda e^{-\lambda\theta}$$

and the exponential distribution function by $\Phi(x)$, that is put

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \lambda \int_{-\infty}^{x} e^{-\lambda\theta} dt.$$

Throughout the paper we use Landau's notation $O(f(n))$ for a term which, when divided by $f(n)$, remains bounded as $n \to \infty$. Similarly, $h(n) = o\big(g(n)\big)$ means that $h(n)/g(n) \to 0$ as $n \to \infty$. Also $h(n) \sim g(n)$ express the fact that $h(n)/g(n) \to 1$ as $n \to \infty$. Thus $h(n) \sim g(n)$ is equivalent to $h(n) - g(n) = o\big(g(n)\big)$.

RESULT

This first lemma represents the continuous case and will be used to prove the next lemma and theorem.

Lemma 1
Let c, θ_0 be the constants defined by (1) and $\lambda > 0$.

Then

*Corresponding author

Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 6, Number 3, July 2022
DOI : https://doi.org/10.33395/sinkron.v7i3.11602
e-ISSN : 2541-2019
p-ISSN : 2541-044X

$\mathbb{P}_c\left(d^{maks} \leq b\sqrt{n} \sim Kc^n\right)$ as $n \to \infty$

where K is the constant

$$e^{\theta_0(b+\theta_0)}\left(\theta_0(\lambda+\theta_0)\right)^{-\frac{1}{2}}.$$

PROOF.
In the continuous, d_i are multivariate distribution which is assumed to have an exponential distribution. The probability is given by

$$\int_{-\infty}^{\infty} \phi(x)\left(\Phi\left(b\sqrt{n}+x\right)/\sqrt{n-2}\right)^n dx$$

Substituting $x = yn$

$$n\int_{-\infty}^{\infty} \phi(x)\left(\Phi\left(b\sqrt{n}+yn\right)/\sqrt{n-2}\right)^n dy = n\int_{-\infty}^{\infty} \phi(x)\left(\Phi\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right)^n dy$$
$$= \lambda n\int_{-\infty}^{\infty} e^{-\lambda ny}\, e^{n\log\Phi\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}} dy$$
$$= \lambda n\int_{-\infty}^{\infty} e^{-n\left[\lambda y-\log\Phi\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right]} dy$$
$$= \lambda n\int_{-\infty}^{\infty} e^{-nf(y)}\, dy$$

where,

$$f(y) = \lambda y - \log\Phi\left(\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right)$$

Defining $R(x) = \frac{\phi(x)}{\Phi(x)}$, we have that $R'(x) = -\left(\lambda + R(x)\right)R(x)$. If $\lambda > 0$ then $R'(x) < 0$.
Now

$$f'(y) = \lambda - R\left(\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right)\sqrt{\frac{n^2}{n-2}}$$
$$f''(y) = 0 - R'\left(\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right)\sqrt{\frac{n^3}{n-2}}$$

So $f''(y) \geq 0$. Therefore, f has a unique minimum, y_0. Furthermore, to calculate the integral result above, an asymptotic expansion is used by expanding the exponential at the stationary point. In this case, $y_0 = \theta_0 + O(n^{-1})$. Then, $f(\theta_0) = f\left(y_0 + O(n^{-2})\right)$ and $f''(\theta_0) \sim -R'(b+\theta_0) = \theta_0(\lambda+\theta_0)$

$$\Lambda n\int_{-\infty}^{\infty} e^{-n\left(f(y_0)+f''(y-y_0)^2/2\right)}\, dy = f''(y_0)^{-\frac{1}{2}} e^{-nf(y_0)}$$
$$\sim f''(\theta_0)^{-\frac{1}{2}} e^{-nf(\theta_0)}$$
$$\sim \left(\theta_0(\lambda+\theta_0)\right)^{-\frac{1}{2}} e^{-n\left(\lambda\theta_0-\log\Phi(b+\theta_0)-\theta_0(b+\theta_0)/n+O\left(n^{-2}\right)\right)}$$

*Corresponding author

Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 6, Number 3, July 2022
DOI : https://doi.org/10.33395/sinkron.v7i3.11602
e-ISSN : 2541-2019
p-ISSN : 2541-044X

$$\sim e^{\theta_0(b+\theta_0)}\left(\theta_0(\lambda+\theta_0)\right)^{-\frac{1}{2}}\Phi(b+\theta_0)e^{-\lambda\theta_0}$$
$$\sim Kc^n$$

Lemma 2

$\lim_{\delta\to 0}\lim_{n\to\infty}\mathbb{P}_c\left(U(\delta)\right)^{1/n} = c.$

Similarly, to the proceeding proof, we need to estimate

$$\int_{-\infty}^{\infty}\phi(x)\left(\Phi\left(b\sqrt{n}+x\right)/\sqrt{n-2}\right)^{n-2\lceil\delta_n\rceil}dx.$$

Proof.

This is monotonic in δ, so for the purposes of taking the limit as $\delta\to 0$, we may replace $n-2\lceil\delta_n\rceil$ by $(1-\delta)n$. The change from the preceding proof is that

$$f(y) = \lambda y-(1-\delta)\log\Phi\left(\left(b+y\sqrt{n}\right)\sqrt{\frac{n}{n-2}}\right).$$

As before from Lemma 1, we have $f''(y)\geq 0$, which ensures that y_0, the unique minimum of f depends continuously on δ. In fact, that $y_0 = \theta_0 + O(\delta+n^{-1})$.

Lemma 3

For $0<\delta_c<\frac{1}{2}$, and there exist ϵ_0,ϵ_1 depends on n,δ_m,δ_s so that $\epsilon_0,\epsilon_1\to 0$, we obtained

$\left(\frac{P_d(U)}{P_c(U)}\right)^{1/n}\to 1$ for $n\to\infty$.

Proof.

Lemma 2 implies that $P_c(U)^{1/n}$ converges to a limit c_1 as $n\to\infty$. Since $U(\delta)$ is decreasing in δ, we have that $c_1\geq c\geq c_0$. Therefore for sufficiently small $\epsilon_0+\epsilon_1(C+\delta_m)$ for C be the constant, we obtained

$$\limsup_{n\to\infty}\mathbb{P}_d(U)^{\frac{1}{2}}\leq c_1\lim_{n\to\infty}e^{\epsilon_0+\epsilon_1(C+\delta_m)}$$
$$\liminf_{n\to\infty}\mathbb{P}_d(U)^{\frac{1}{2}}\geq c_1\lim_{n\to\infty}e^{\epsilon_0+\epsilon_1(C+\delta_m)}$$

Letting $\delta_s\to 0$ then $\delta_m\to 0$ show that $\mathbb{P}_d(U)^{\frac{1}{n}}\to c_1$ as $n\to\infty$. Since c_1 was also the limit of $\mathbb{P}_d(U)^{\frac{1}{n}}$ as $n\to\infty$, the lemma is proved. $\mathbb{P}_c(U)^{\frac{1}{n}} = \mathbb{P}_d(U)^{\frac{1}{n}}$

Theorem 4

Let $0<p<1$, $q=1-p$ and b be fixed. For G a random graph from $\mathcal{G}(n,p)$, let

$$p_n = \mathbb{P}\left(d^{maks}(G)\leq np+b\sqrt{npq}\right)$$

*Corresponding author

Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 6, Number 3, July 2022
DOI : https://doi.org/10.33395/sinkron.v7i3.11602
e-ISSN : 2541-2019
p-ISSN : 2541-044X

$p_n^{1/n} \to c$ as $n \to \infty$,

where c is a constant depending only on b and $\lambda > 0$, given by

c =maks$\{\Phi(b+\theta)e^{-\lambda\theta}\}$

Proof.

From Lemma 2 we get $\mathbb{P}_d(U)^{\frac{1}{n}} \to c$. The events U_i and event U are down-sets when considered as subsets of $G(n,p)$. This implies that $p_n \geq q\mathbb{P}_d(U)\mathbb{P}_d(U_1)^{2r_0}$. The central limit theorem tell us that $\mathbb{P}_d(U_1) \to \Phi(b)$ as $n \to \infty$. We get

$$\lim_{n\to\infty} p_n^{\frac{1}{n}} \geq \lim_{n\to\infty} \mathbb{P}_d(U)^{1/n} = c$$

The above maximum is attained at $\theta_0 = \theta(b)$ the root of equation

$$\theta_0\Phi(b+\theta_0) = \phi(b+\theta_0)$$

Let $p = q = \frac{1}{2}$, $b = 0$ and $\lambda = 0.5$ and $\theta_0 = 0.506054468$ we get that $c = 0.595656764$

REFERENCES

Bhamidi, S., Chakraborty, S., Cranmer, S. & Desmarais, B. (2018). Weighted exponential random graph models: Scope and large network limits. *Journal of Statistical Physics*, *173*(3), 704–735.

Bollobás, B. (1981). Degree sequences of random graphs. *Discrete Mathematics*, *33*(1), 1–19.

Bollobás, B. (1998). Random graphs. In *Modern graph theory* (pp. 215–252). Springer.

Courant, R. & Hilbert, D. (1954). Methods of mathematical physics. *Bulletin of the American Mathematical Society*, *60*, 578–579.

Erdos, P. & Rényi, A. (1960). On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, *5*(1), 17–60.

Feller, W. (2008). *An introduction to probability theory and its applications*. John Wiley & Sons.

Grimmett, G. & Stirzaker, D. (2020). *Probability and random processes*. Oxford university press.

Kleitman, D. J. (1966). Families of non-disjoint subsets. *Journal of Combinatorial Theory*, *1*(1), 153–155.

McKay, B. D. & Wormald, N. C. (1997). The degree sequence of a random graph. I. The models. *Random Structures & Algorithms*, *11*(2), 97–117.

Riordan, O. & Selby, A. (2000). The maximum degree of a random graph. *Combinatorics, Probability and Computing*, *9*(6), 549–572.

*Corresponding author