

# Mathematical Modeling of the Vehicle Routing Problem with Relaxed Time Windows and Delay Penalties

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Submitted : Jun 30, 2025 | Accepted : Aug 15, 2025 | Published : Aug 17, 2025

**Abstract:** The Vehicle Routing Problem with Relaxed Time Windows (VRP-RTW) is an extension of the classic Vehicle Routing Problem (VRP) that incorporates flexibility in service time windows. In VRP-RTW, vehicles are allowed to arrive later than the specified time window. However, a violation will be imposed for exceeding the specified time limit. in the form of fines or similar penalties. This research aims to design a mathematical model for VRP-RTW to minimize total travel costs and delay penalties, while ensuring that all customers are served within the capacity limits of the available vehicles. This research uses literature review methods and mathematical formulation approaches to describe the logistics distribution problem. The developed model considers several constraints, such as vehicle capacity, route balance, and service time limitations. The results of this research are expected to contribute to more efficient and flexible logistics distribution decision-making and serve as a basis for the development of vehicle route optimization models that can be applied in real-world scenarios.

**Keywords:** Vehicle Routing Problem; Relaxed Time Windows; Delay Penalties; Logistics Distribution; Combinatorial Optimization

## INTRODUCTION

Logistics is a sequence of events that include the planning, implementation, and oversight of transferring commodities, services, energy, or other resources from the origin to the end user. The objective of these actions is to optimize capital utilization, striving to achieve equilibrium by decreasing costs while preserving service quality and customer happiness (Dadzie & Richard, 2025; Rakyta et al., 2022). Nonetheless, improvements in information technology facilitating the swift and nearly costless conveyance of information have augmented the complexity and significance of logistics. In contrast to conventional distribution systems that necessitated the delivery of items to warehouses or retail outlets, consumers now favor home delivery of parcels (Jazemi et al., 2023). The logistics system comprises five primary components: facility location structure, transportation, inventory, communication, and commodities handling and storage (Golabek et al., 2021).

This challenge involves a set of vehicle routes, typically variable, that will efficiently service a designated number of specified customers while adhering to various constraints. The Vehicle Routing Problem (VRP) identifies the ideal path to minimize travel expenses from a depot to clients situated in diverse locations with fluctuating demand levels (Sitompul & Horas, 2021). The advancement of transportation will enhance human mobility, production resources, and the distribution of commercial goods. Increased mobility enhances the speed of movement and equipment use, resulting in more efficient distribution and reduced time for processing and relocating resources from less advantageous areas to more beneficial ones. Consequently, increased mobility correlates with elevated productivity levels (Voigt et al., 2023).

The solution to the Vehicle Routing Problem (VRP) consists of routes that must be taken by vehicles to meet all customer demands, where each route will be traveled by a single vehicle starting and ending at the depot. Vehicle Routing Problem (VRP) is one of the most studied combinatorial optimization problems and is related to the planning of optimal routes to be used by a fleet of vehicles to serve a number of customers (Chai et al., 2022). To find the best route, the distance and time obtained must also be minimized and reached as quickly as possible to make the fuel/gasoline used more efficient (Toth & Vigo, 2002). There are several variants of the VRP, such as the Vehicle Routing Problem with Time Windows (VRPTW) (Chai et al., 2022), Vehicle Routing Problem with

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Pick-Up and Delivery (VRPPD) (Dethloff, 2001), and Capacitated Vehicle Routing Problem (CVRP) (Ramadhani et al., 2019). The vehicle routing problem with time windows is a widely recognized issue in the logistics field, which involves optimizing delivery routes for a fleet of vehicles while considering specific time windows for each delivery.

In the Vehicle Routing Problem with Time Windows (VRPTW), each client must be serviced within a designated time frame. Time limits emerge from product limitations or production restrictions or may be imposed by customers due to their inventory regulations. In addition to client time limitations, journey duration encompasses transit time between all customer locations or between customers and the central depot. Vehicles must deliver services to clients within a designated timeframe at minimal expense. Vehicles may arrive at the customer prior to the designated time frame but must remain on standby if they arrive before the customer is prepared to be served; late arrivals after the conclusion of the time period are prohibited (Mawengkang et al., 2024). Constrained Vehicle Routing Problem Time Windows (VRPTW) has attracted considerable interest from the research community owing to its various real-world uses. Examples include waste collection, bus route optimization, and specialized bus services (Liu et al., 2023). Routes must be structured to ensure that each location is visited solely once by a single vehicle within a certain time frame, with all routes commencing and concluding at the depot, and the cumulative demand of all locations on a given route must not surpass the vehicle's capacity (Bräysy & Gendreau, 2005).

VRPTW can be mathematically expressed as follows:

$$\min \sum_{k \in V} \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ijk} \quad (1)$$

Constraint:

$$\sum_{k \in V} \sum_{j \in N} x_{ijk} = 1 \quad \forall i \in C \quad (2)$$

$$\sum_{i \in C} d_i \sum_{j \in N} x_{ijk} \leq q \quad \forall k \in V \quad (3)$$

$$\sum_{j \in N} x_{0jk} = 1, \forall k \in V \quad (4)$$

$$\sum_{i \in N} x_{ihk} - \sum_{j \in N} x_{hjk} = 0, \forall h \in C, \forall k \in V \quad (5)$$

$$\sum_{i \in N} x_{i,n+1,k} = 1, \forall k \in V \quad (6)$$

$$s_{ik} + t_{ij} - K(1 - x_{ijk}) \leq s_{jk}, \forall i, j \in N, \forall k \in V \quad (7)$$

$$a_i \leq s_{ik} \leq b_i, \forall i \in N, \forall k \in V \quad (8)$$

$$x_{ijk} \in \{0, 1\} \quad \forall i, j \in N, \forall k \in V \quad (9)$$

The objective minimizes the fleet's total travel cost by summing the cost  $c_{ij}$  of each arc  $i \rightarrow j$  whenever the binary variable  $x_{ijk}$  indicates that vehicle  $k$  traverses that arc. In equation (2), all customers are visited only once, and equation (3) ensures that vehicles do not carry loads exceeding the maximum capacity. Then, equations (4), (5), and (6) explain that each vehicle must leave depot 0, after which the vehicle arrives at the customer and departs to another destination, and finally, all vehicles must return to the depot  $n+1$ . In equation (7), vehicle  $k$  cannot arrive at  $j$  before  $s_{ik} + t_{ij}$  if the vehicle departs from  $i$  to  $j$  with  $K$  being a scalar value. After that, equation (8) explains that there are time windows and equation (9) represents the constraint of an integral. Therefore, unused vehicles are modeled by driving an empty route  $(0, n+1)$  [10]. The Vehicle Routing Problem Relaxed Time Windows (VRP-RTW) is a part of the classic Vehicle Routing Problem (VRP) that involves delivering goods to customers, where vehicles must visit customers within a certain time frame known as time windows. However, in this problem, the time windows are relaxed, meaning there is some flexibility regarding early or late arrivals. However, there is a penalty imposed if the vehicle arrives outside the specified time window. Thus, in this research, the author aims to develop a model of the Vehicle Routing Problem with Relaxed Time Windows (VRPRTW) in logistics to determine the optimal routing by minimizing the total travel cost that the vehicle will incur to serve customers, thereby providing a more accurate solution.

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## METHOD

This research adopts a literature review method by utilizing various references and collecting relevant journals and books to serve as the foundation for the study. The steps carried out in this research are as follows:

1. Identifying the research topic – the author identifies the topic to be studied as well as the issues related to the topic.
2. Literature review – the author collects and studies various previous research sources related to the topic as part of the literature study, which also serve as references for this research.
3. Determining the notations, parameters, and decision variables to be used in this research.
4. Formulating the objective function to minimize travel costs and penalties for time window violations. The penalty refers to additional costs incurred when a vehicle arrives after the latest time window, and penalties may also be applied for early arrivals using a different penalty function (Watling, 2006).
5. Defining the constraint functions, including:
  - a. Customer visits exactly once.
  - b. Vehicle flow balances in and out of each node.
  - c. Vehicles depart from and return to the depot.
  - d. Vehicle capacity constraint.
  - e. Service time window constraint with relaxation.
  - f. Time relationship between nodes.
  - g. Decision variable constraints.
6. Explaining the resulting model in a manner that is easy to understand.

## RESULT

The problem discussed in this research is a development of the Vehicle Routing Problem (VRP), which is an optimization problem aimed at determining the most optimal routes for a number of vehicles in distributing goods from a central depot to several customers spread across various locations. This research discusses the Vehicle Routing Problem Relaxed Time Windows with Penalty (VRP-RTW-P), where each customer has a specific service time window, but vehicles are allowed to arrive outside of these time windows with the consequence of a penalty imposed for delays or early arrivals. The main objective of this problem is to obtain an optimal routing with limited capacity while minimizing the total travel cost, ensuring that all customers are served by considering the relaxation of the given time windows, with penalties for violations.

The variables and parameters used in the mathematical model of VRP-RTW for the routing problem are as follows:

### Notation:

$N$  : Set of customers including the depot

$K$  : Set of available vehicles

### Parameters:

$c_{ij}$  : Cost or travel distance from node  $i$  to node  $j$

$d_i$  : Demand of customer  $i$

$Q_k$  : Capacity of vehicle  $k$

$s_i$  : Service time at node  $i$

$t_i$  : Arrival time of the vehicle at node  $i$

$M$  : A large number (Big-M) used to activate constraints only if  $x_{ijk} = 1$

$\alpha_i$  : Penalty coefficient for service time lateness

$t_{ij}$  : Travel time from node  $i$  to node  $j$

### Decision Variables:

$x_{ijk}$  : Binary variable with value : 1, if the vehicle passes the route  $i$  to  $j$  and 0, otherwise

$a_i, b_i$  : Lower and upper bounds of the service time window for customer  $i$

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$P_i$  : Penalty if the service is provided after the upper limit  $b_i$

Mathematical Model

$$\min Z = \sum_{(i,j) \in N} c_{ij} x_{ijk} + \sum_{i \in N} \alpha_i P_i \quad (10)$$

Constraints:

$$\sum_{k \in K} \sum_{j \in N, j \neq i} x_{ijk} = 1 \quad \forall i \in N \quad (11)$$

$$\sum_{j \in N} x_{jik} = \sum_{j \in N} x_{ijk} \quad \forall i \in N, \forall k \in K \quad (12)$$

$$\sum_{j \in N, j \neq 0} x_{0jk} = 1, \quad \forall k \in K \quad (13)$$

$$\sum_{i \in N, i \neq 0} x_{i0k} = 1, \quad \forall k \in K \quad (14)$$

$$\sum_{i \in N} d_i \sum_{j \in N} x_{ijk} \leq Q_k \quad \forall k \in K \quad (15)$$

$$a_i \leq t_i \leq b_i \quad \forall i \in N \quad (16)$$

$$P_i \geq t_i - b_i \quad \forall i \in N \quad (17)$$

$$t_j \geq t_i + s_i + t_{ij} - M(1 - \alpha_i x_{ijk}), \quad \forall i, j \in N, k \in K \quad (18)$$

$$x_{ijk} \in \{0, 1\}, \quad \forall i, j \in N$$

$$P_i \geq 0 \quad (19)$$

$$t_i \geq 0$$

The objective minimizes the overall cost by adding up the travel expense on every selected arc  $c_{ij}x_{ij}$  and the customer-specific penalty  $\alpha_i P_i$  incurred for any service-time violation. In constraint equation (11) each customer must be visited exactly once, and in constraint equation (12) it must be ensured that the vehicle enters and exits each node in a balanced manner. Then, in constraint equation (13) each vehicle must start and end at the depot. Finally, in constraint equation (14) the total customer demand served by the vehicle must not exceed its capacity. In equation (15) this equation ensures that the service starts within the allowed time frame, allowing for relaxation with the associated penalty. Equation (16) maintains the order of visit times between nodes while allowing for delays with penalties. The last equation, (17), is the constraint on the decision variables.

## DISCUSSION

Based on the defined problem, the Vehicle Routing Problem Relaxed Time Windows model aims to minimize the total travel cost while ensuring that all customers are served by taking into account the relaxation in the given time windows, with penalties for violations.

This model also uses several constraint functions where the constraint functions are as follows: The objective minimizes the overall cost by adding up the travel expense on every selected arc  $c_{ij}x_{ij}$  and the customer-specific penalty  $\alpha_i P_i$  incurred for any service-time violation. In constraint equation (11) each customer must be visited exactly once, and in constraint equation (12) it must be ensured that the vehicle enters and exits each node in a balanced manner. Then, in constraint equation (13) each vehicle must start and end at the depot. Finally, in constraint equation (14) the total customer demand served by the vehicle must not exceed its capacity. In equation (15) this equation ensures that the service starts within the allowed time frame, allowing for relaxation with the associated penalty. Equation (16) maintains the order of visit times between nodes while allowing for delays with penalties. Equation (17) each vehicle has a daily operational time limit to avoid excessive workload. In equation (18) arrange for each vehicle to follow the maximum route limits determined based on its type, as a form of vehicle variation constraint. Equation (19) if the duration of the vehicle's journey exceeds a certain time limit, the vehicle is required to stop to rest. The last equation, (20) is the constraint on the decision variables.

Despite offering a flexible approach to vehicle routing under relaxed time windows, this study has several limitations. First, the model assumes deterministic travel times and demands, which may not fully capture real-world uncertainties such as traffic congestion, vehicle breakdowns, or sudden demand fluctuations. Second, the penalty structure is simplified as a linear cost, whereas in practice, penalties may be non-linear or negotiated

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differently depending on customer contracts. Third, the research relies on a mathematical formulation without computational experiments, so the scalability and efficiency of the model on large instances remain untested. Lastly, factors such as heterogeneous vehicle fleets, driver regulations, and multi-depot configurations were not considered, which restricts the generalizability of the findings. These limitations open opportunities for future research to extend the model toward more dynamic, stochastic, and realistic logistics environments.

### CONCLUSION

This research discusses the Vehicle Routing Problem with Relaxed Time Windows (VRP-RTW) as a form of development of the classical Vehicle Routing Problem (VRP) that considers time flexibility in the distribution process. This research designs a mathematical model that not only considers the minimum travel cost but also takes into account the penalties arising from violations of customer service time windows. Based on the developed model formulation, it was found that the VRP-RTW approach allows vehicles to arrive earlier or later than the specified time window, provided that any violation of the time windows will incur a penalty. This provides flexibility in the distribution process, as well as increasing the chances of obtaining a feasible and operationally efficient solution. Overall, this research makes a significant contribution to the development of optimization models for modern logistics, particularly in the context of vehicle routing scheduling that is time-flexible yet cost-optimal.

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